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MULTIPLICITIES AND MINIMAL WIDTHS FOR (0, 1)-MATRICES

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. This includes the study of combinatorial problems, with applications to communication networks, switching circuits, error-detecting and error-correcting codes, etc.

These three publications report on a study of

A number of these combinatorial problems can be formulated in terms of matrices made up of columns of zeros and ones. In the present Memorandum the authors continue the work of RM-2896-PR, <u>Widths and Heights of (0, 1)-Matrices</u>, to obtain a simple construction that produces a matrix having a minimal α-width for all α.

The work of the coauthor, Dr. Ryser, was supported in part by the Office of Ordnance Research.

SUMMARY

In a previous Memorandum (RM-2896-PR) the notion of the α-width of a (0, 1)-matrix was introduced, and a formula for the minimal α-width taken over the class of all (0, 1)-matrices having specified row and column sums was obtained. The present Memorandum continues the study begun in RM-2896-PR. The principal new result is a simple construction that produces a matrix having the property that its α-widths are minimal for all α.

CONTENTS

MULTIPLICITIES AND MINIMAL WIDTHS FOR (0, 1)-MATRICES

1. INTRODUCTION

In a previous paper [1] the notion of the α-width $\varepsilon_A(\alpha)$ of a (0, 1)-matrix A was introduced, and a formula for the minimal α-width $\tilde{\varepsilon}(\alpha)$, taken over the class $\mathfrak{A}$ of all (0, 1)-matrices having the same row and column sums as A, was obtained. The main tool in arriving at this formula was a block decomposition theorem (Theorem 3.1 of [1], repeated in this paper as Theorem 3.1) that established the existence, in the class $\mathfrak{A}$ generated by A, of certain matrices having a simple block structure. The block decomposition theorem does not itself directly involve the notion of minimal α-width, but rather centers around a related class concept, that of multiplicity. We review both of these notions in Sec. 3, together with some other pertinent definitions and results.

The present paper continues the study begun in [1]. The principal contribution is a simple construction which produces a single matrix $\tilde{A}$ in the class $\mathfrak{A}$ that has some remarkable properties: first, the partial row-sum vectors of $\tilde{A}$ are as smooth as possible in the sense of majorization (Theorem 4.2); second, all minimal α-widths and multiplicities for the class $\mathfrak{A}$ can be obtained directly from $\tilde{A}$ (Theorem 4.3 and Corollary 4.4).

In the concluding section we apply the matrix $\tilde{A}$ in the solution of a problem closely related to the minimal width problem. For each A in $\mathfrak{A}$, define $\mu_A(\beta)$ to be the maximal number of columns that can be selected from A in such a way

that the resulting submatrix has at most β 1's in each row. It follows readily that the sequences $\varepsilon_A(\alpha)$ and $\mu_{A'}(\beta)$, where A' is the complement of A, determine each other, and hence that the class sequence $\bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta)$ is determined by the minimal width sequence for the complementary class.

2. A BASIC CONSTRUCTION

Let A be a matrix of m rows and n columns whose entries are either 0 or 1. We call A a (0, 1)-matrix of size m by n. Let the sum of row i of A be denoted by r_i and the sum of column j of A by s_j. We call

$$(2.1) \qquad R = (r_1, r_2, \ldots, r_m)$$

the row-sum vector of A, and

$$(2.2) \qquad S = (s_1, s_2, \ldots, s_n)$$

the column-sum vector of A. These vectors determine a class,

$$(2.3) \qquad \mathfrak{A} = \mathfrak{A}(R, S),$$

consisting of all (0, 1)-matrices of size m by n having row-sum vector R and column-sum vector S. Simple necessary and sufficient conditions on R and S are known in order that the class $\mathfrak{A}(R, S)$ be nonempty [3], [6].

Let A be in $\mathfrak{A}$ and consider the 2 by 2 submatrices of A of the types

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An <u>interchange</u> is a transformation of the elements of A that changes a minor of one of these types into the other, leaving all other elements fixed. The interchange theorem [6] asserts that if A and B are in $\mathfrak{A}$, then A is transformable into B by interchanges.

Throughout this paper we suppose, without loss of generality, that $\mathfrak{A}$ is nonempty and that

$$(2.4) \qquad r_1 \geq r_2 \geq \cdots \geq r_m > 0,$$

$$(2.5) \qquad s_1 \geq s_2 \geq \cdots \geq s_n > 0.$$

Such an $\mathfrak{A}$ is termed <u>normalized</u>.

Let $A = [a_{ij}]$ be in $\mathfrak{A}$. We call the column vector

$$(2.6) \qquad R_\varepsilon = \begin{bmatrix} \sum_{j=1}^{\varepsilon} a_{1j} \\ \sum_{j=1}^{\varepsilon} a_{2j} \\ \vdots \\ \sum_{j=1}^{\varepsilon} a_{mj} \end{bmatrix}$$

the ε^{th} <u>partial row-sum vector of</u> A. Thus $R_n = R^T$, where R^T denotes the transpose of R.

Given the vectors R and S for a normalized class $\mathfrak{A}$, there is a simple rule for constructing an A in $\mathfrak{A}$. This rule may be stated somewhat loosely as follows: <u>Select any column</u> j <u>and insert its 1's in the positions corresponding to the</u> s_j <u>largest</u>

<u>row sums</u>. <u>Delete column</u> j, <u>reduce each of these</u> s_j <u>row sums</u> <u>by</u> 1, <u>and repeat the entire procedure on another column</u>.

<u>Example</u>. Let $\mathfrak{A}$ be determined by

$$R = (7, \; 6, \; 3, \; 2, \; 2, \; 2, \; 2, \; 2),$$
$$S = (4, \; 4, \; 4, \; 4, \; 4, \; 4, \; 1, \; 1).$$

Suppose we apply the rule from "right to left," i.e., select the last column first, then the next to last, and so on, and give preference to the bottommost positions in a column in case of ties (this keeps the partial row sums monotone). The rule then constructs the following matrix $\tilde{A}$, having partial row-sum vectors given by the matrix $\tilde{M}$:

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\tilde{M} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

The validity of the construction can be established by a simple interchange argument, as follows. Let A be in $\mathfrak{A}$ and suppose B has been constructed by the rule. We must show that B is in $\mathfrak{A}$. Assume that in constructing B, the 1's in column j were assigned initially; we may suppose without loss of generality that these 1's occupy the s_j topmost positions in column j. If A has a 0 above a 1 in column j, then by the monotonicity of the row sums of A, there is an interchange that switches this 0 and 1 in column j. Hence we may apply interchanges involving column j of A to obtain a transformed matrix in $\mathfrak{A}$ whose j^{th} column agrees with the j^{th} column of B. We can now suppress column j of the transformed A and B, and repeat the argument. Eventually A has been transformed by interchanges into B, and thus B is in $\mathfrak{A}$.

Of course an analogous procedure in which the roles of rows and columns are reversed also constructs a matrix in the class.

In Sec. 4 we shall apply this construction in the right to left order (as in the example), giving preference to bottommost positions in a column in case of ties (as in the example). The resulting matrix will be denoted by $\tilde{A}$.

3. A REVIEW OF MULTIPLICITY AND WIDTH

Let $\mathfrak{A} = \mathfrak{A}(R, S)$ be a normalized class and let α and ε be integers satisfying

$$(3.1) \qquad 0 \leq \alpha \leq r_m,$$

$$(3.2) \qquad 1 \leq \varepsilon \leq n.$$

We say that a pair α, ε in the respective ranges (3.1), (3.2) are *compatible* if there is an A in $\mathfrak{A}$ having an m by ε submatrix E^* each of whose row sums is at least α. If α and ε are compatible, consider the class of all m by ε submatrices E^* of the matrices A in $\mathfrak{A}$ with the row sums of E^* at least α, and let δ^* denote the number of rows of E^* whose sums are precisely α. The nonnegative integer

$$(3.3) \qquad \delta = \delta(\alpha, \varepsilon)$$

that is equal to the minimum of the integers δ^* is called the *multiplicity* of α with respect to ε.

In [1] the following theorem was proved.

Theorem 3.1. Let α be compatible with ε and of multiplicity δ with respect to ε. Then there is a matrix A in the normalized $\mathfrak{A}$ of the form

$$(3.4) \qquad A = \left[\begin{array}{c|c|c} M & J & X \\ \hline F & & \\ \cline{1-1} E & \raisebox{1.5ex}{Y} & \raisebox{1.5ex}{O} \end{array}\right].$$

Here E is of size δ by ε with exactly α 1's in each row. M is a matrix of size e by ε with $\alpha + 1$ or more 1's in each row. F is a matrix of size $m - (e + \delta)$ by ε with exactly $\alpha + 1$ 1's in each row. J is a matrix of 1's of size e by $f - \varepsilon$ and O is a zero matrix. The degenerate cases $e = 0$, $e + \delta = m$, $\delta = 0$, $f = \varepsilon$, and $f = n$ are not excluded.

The α-<u>width</u> $\varepsilon_A(\alpha)$ of a matrix A in the normalized $\mathfrak{A}$ is the fewest number of columns that can be selected from A so that the resulting submatrix E* has row sums at least α. Here $1 \leq \alpha \leq r_m$. Then

$$(3.5) \qquad \tilde{\varepsilon}(\alpha) = \min_{A \text{ in } \mathfrak{A}} \varepsilon_A(\alpha)$$

is the <u>minimal</u> α-<u>width</u> of $\mathfrak{A}$. The integer $\tilde{\varepsilon}(\alpha)$, which can also be described as the least ε compatible with α, has been explicitly determined in terms of the vectors R and S in [1]. This determination used the function

$$(3.6) \qquad N(\varepsilon, e, f) = r_{e+1} + \dots + r_m - (s_{\varepsilon+1} + \dots + s_f) + e(f - \varepsilon),$$

where ε, e, f are integer parameters satisfying

$$(3.7) \qquad 0 \leq \varepsilon \leq n,$$

$$(3.8) \qquad 0 \leq e \leq m,$$

$$(3.9) \qquad \varepsilon \leq f \leq n.$$

Precisely, $\tilde{\varepsilon}(\alpha)$ is the first ε such that

$$(3.10) \qquad N(\varepsilon, e, f) \geq \alpha(m - e)$$

for all e, f satisfying (3.8), (3.9). Note that if A is in $\mathfrak{A}(R, S)$ and if we write

$$(3.11) \qquad A = \begin{bmatrix} * & Y & * \\ X & * & Z \end{bmatrix},$$

with X of size $(m - e)$ by ε and Y of size e by $(f - \varepsilon)$, then

$$(3.12) \qquad N(\varepsilon, e, f) = N_1(X) + N_0(Y) + N_1(Z),$$

where $N_1(Q)$ $[N_0(Q)]$ denotes the number of 1's (0's) in a (0, 1)-matrix Q.

It was also proved in [1] that if α, ε are compatible, then

$$(3.13) \quad \delta(\alpha, \varepsilon) = (\alpha + 1)m - s_\varepsilon - \min_{\substack{0 \le e \le m \\ \varepsilon \le f \le n}} [N(\varepsilon - 1, e, f) + \alpha e].$$

(Only the special case of (3.13) in which $\varepsilon = \tilde{\varepsilon}(\alpha)$ is stated explicitly in [1], but the proof there establishes (3.13) in general.)

4. THE MATRICES $\breve{A}$ AND $\breve{M}$

If

$$T = (t_1, t_2, \ldots, t_m),$$

$$T^* = (t_1^*, t_2^*, \ldots, t_m^*),$$

are two vectors of nonnegative integers, then T is <u>majorized</u> by T* [4], [5], and we write

$$(4.1) \qquad T \prec T^*,$$

provided that, with subscripts renumbered,

$$(4.2)\quad t_1 \geq t_2 \geq \cdots \geq t_m, \quad t_1^* \geq t_2^* \geq \cdots \geq t_m^*,$$

$$(4.3)\quad t_1 + t_2 + \cdots + t_e \leq t_1^* + t_2^* + \cdots + t_e^*, \quad (e = 1, 2, \ldots, m-1),$$

$$(4.4)\quad t_1 + t_2 + \cdots + t_m = t_1^* + t_2^* + \cdots + t_m^*.$$

In connection with this concept, we prove the following lemma, which will be used in the proof of Theorem 4.2.

Lemma 4.1. _Let_ $T = (t_1, t_2, \ldots, t_m)$ _and_ $T^* = (t_1^*, t_2^*, \ldots, t_m^*)$ _be two vectors of nonnegative integers satisfying_ (4.2), (4.3), (4.4). _Let_ U _be obtained from_ T _by reducing_ k _of its positive components in positions_ $i_1, i_2, \ldots, i_k$ _by_ 1. _Similarly let_ U^* _be obtained from_ T^* _by reducing_ k _of its positive components in positions_ $j_1, j_2, \ldots, j_k$ _by_ 1. _If_ $i_1 \leq j_1, i_2 \leq j_2, \ldots, i_k \leq j_k$, _then_ $U \prec U^*$.

Proof. We proceed by induction on k. Let $k = 1$ and set $i_1 = i$, $j_1 = j$. We may take U and U^* to be monotone nonincreasing by assuming that component i' of T has been reduced by 1 to get U, component j' of T^* has been reduced by 1 to get U^*. Here $i' \geq i$, $j' \geq j$ and

$$(4.5)\quad t_i = t_{i+1} = \cdots = t_{i'} > t_{i'+1},$$

$$(4.6)\quad t_j^* = t_{j+1}^* = \cdots = t_{j'}^* > t_{j'+1}^*,$$

where $t_{i'+1} = 0$ if $i' = m$ and $t^*_{j'+1} = 0$ if $j' = m$. If now $i' \leq j'$, then clearly $U \prec U^*$. Suppose that $i' > j'$. Then

(4.7) $$1 \leq i \leq j \leq j' < i' \leq m.$$

If U is not majorized by U^*, there is an integer e satisfying

(4.8) $$j' \leq e < i'$$

for which

(4.9) $$t_1 + t_2 + \ldots + t_e = t^*_1 + t^*_2 + \ldots + t^*_e.$$

By assumption,

(4.10) $$t_1 + t_2 + \ldots + t_{e-1} \leq t^*_1 + t^*_2 + \ldots + t^*_{e-1},$$

(4.11) $$t_1 + t_2 + \ldots + t_{e+1} \leq t^*_1 + t^*_2 + \ldots + t^*_{e+1}.$$

Subtracting (4.10) from (4.9), and (4.9) from (4.11), yields

(4.12) $$t_e \geq t^*_e,$$

(4.13) $$t_{e+1} \leq t^*_{e+1}.$$

By (4.5), (4.7), (4.8), we have

(4.14) $$t_e = t_{e+1}.$$

Thus (4.12), (4.13), (4.14) and $t^*_e \geq t^*_{e+1}$ imply

(4.15) $$t^*_e = t_e = t^*_{e+1}.$$

If $e = j'$, this contradicts (4.6). If, on the other hand, $j' < e$, then from (4.15) and (4.9) we have

$$(4.16) \qquad t_1 + t_2 + \dots + t_{e-1} = t_1^* + t_2^* + \dots + t_{e-1}^*.$$

We may now repeat the argument with $e - 1$ in place of e. Eventually (4.6) is contradicted. This verifies Lemma 4.1 for $k = 1$.

Assume the validity of the lemma for $k - 1$. Let P and P^* be obtained from T and T^* by reducing components $i_2, i_3, \dots, i_k$ of T and components $j_2, j_3, \dots, j_k$ of T^*. By the induction assumption, we have $P \prec P^*$. Of course P and P^* may not be in monotone nonincreasing order, but such rearrangements of them can be secured without disturbing the i_1 position of P or the j_1 position of P^*. Applying the argument used for $k = 1$ to these rearrangements, we see that $U \prec U^*$, thus proving Lemma 4.1.

Let the vectors R and S be given for a normalized class $\mathfrak{A}(R, S)$ and let $\breve{A}$ denote the matrix in $\mathfrak{A}(R, S)$ constructed by the rule of Sec. 2, proceeding column-wise from right to left and giving preference within a column to bottommost positions in case of ties. We now prove

Theorem 4.2. Let A be arbitrary in the normalized class $\mathfrak{A}$ and let A have partial row-sum vectors, $R_1, R_2, \dots, R_n$. Let the matrix $\breve{A}$ in $\mathfrak{A}$ have partial row-sum vectors $\breve{R}_1, \breve{R}_2, \dots, \breve{R}_n$. Then $\breve{R}_\varepsilon \prec R_\varepsilon$, $\varepsilon = 1, 2, \dots, n$.

<u>Proof</u>. We prove Theorem 4.2 by induction. Note that

$$\tilde{R}_n = R_n = R^T, \tag{4.17}$$

and hence the theorem is valid with $\varepsilon = n$. Assume that

$$\tilde{R}_{\varepsilon+1} \prec R_{\varepsilon+1}, \tag{4.18}$$

and consider the vectors $\tilde{R}_\varepsilon$, R_ε. The vector R_ε is obtained from a nonincreasing rearrangement $R^*_{\varepsilon+1}$ of $R_{\varepsilon+1}$ by reducing $s_{\varepsilon+1}$ distinct components of $R^*_{\varepsilon+1}$ by 1. A rearrangement of $\tilde{R}_\varepsilon$ is obtained from the monotone $\tilde{R}_{\varepsilon+1}$ by reducing the first $s_{\varepsilon+1}$ components of $\tilde{R}_{\varepsilon+1}$ by 1. By Lemma 4.1, we have

$$\tilde{R}_\varepsilon \prec R_\varepsilon . \tag{4.19}$$

This proves Theorem 4.2.

<u>Theorem 4.3</u>. <u>The matrix</u> $\hat{A}$ <u>is of form</u> (3.4) <u>for all compatible pairs</u> α, ε.

<u>Proof</u>. Let α and ε be compatible, and let

$$A = \left[\begin{array}{c|c|c} M & J & X \\ \hline F & & \\ \cline{1-1} E & \smash{\raisebox{1.5ex}{Y}} & \smash{\raisebox{1.5ex}{0}} \end{array}\right] \tag{4.20}$$

be the matrix whose existence is given by Theorem 3.1. Thus E is of size $\delta = \delta(\alpha, \varepsilon)$ by ε, with exactly α 1's in each row; M is of size e by ε, with at least $(\alpha + 1)$ 1's in each row;

F is of size $m - (\delta + e)$ by ε with exactly $(\alpha + 1)$ 1's in each row; J is a matrix of size e by $(f - \varepsilon)$, consisting entirely of 1's; and O is a zero matrix.

Consider the first ε columns of $\tilde{A}$. Each of the row sums of these ε columns must be at least α, for otherwise we may use the matrix A to contradict $\tilde{R}_\varepsilon \prec R_\varepsilon$. By the definition of multiplicity, the first ε columns of $\tilde{A}$ cannot have fewer than δ rows with exactly α 1's in each row. Nor can these ε columns have more than δ rows with exactly α 1's in each row. For if this were the case, again $\tilde{R}_\varepsilon \prec R_\varepsilon$ would be contradicted. Hence, since $\tilde{R}_\varepsilon$ is monotone, $\tilde{A}$ has a δ by ε matrix of form E in the lower left corner, and the portion of $\tilde{A}$ corresponding to M and F of A must contain at least $(\alpha + 1)$ 1's in each row. But

$$N(\varepsilon, e, f) = N_1(F) + N_1(E) + N_0(J) + N_1(O)$$

is a class invariant. Hence the portions of $\tilde{A}$ corresponding to F, J, and O of A are of the desired form. This completes the proof.

Define $\tilde{M}$ to be the m by n matrix of nonnegative integers whose column vectors are the partial row-sum vectors of $\tilde{A}$,

$$\tilde{M} = [\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n]. \tag{4.21}$$

We call $\tilde{M}$ the <u>multiplicity matrix</u> of the normalized class $\mathfrak{A}$. Corollary 4.4 collects some immediate consequences of Theorem 4.3 that justify this nomenclature.

Corollary 4.4. Let $\tilde{M} = [\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n]$ *be the multiplicity matrix of the normalized class* $\mathfrak{A}$. *Then* α *and* $\mathcal{E}$ *are compatible if and only if the last component of* $\tilde{R}_{\mathcal{E}}$ *is at least* α. *If* α *and* $\mathcal{E}$ *are compatible, the multiplicity* $\delta(\alpha, \mathcal{E})$ *of* α *with respect to* $\mathcal{E}$ *is equal to the number of components of* $\tilde{R}_{\mathcal{E}}$ *that are equal to* α.

Example. To illustrate Theorem 4.3, consider the example of Sec. 2 corresponding to the compatible pair $\alpha = 1$, $\mathcal{E} = 3$:

$$\tilde{A} = \left[\begin{array}{ccc|ccc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right],$$

$$\tilde{M} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 6 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}.$$

From $\tilde{M}$, the multiplicity $\delta(\alpha, \varepsilon)$ for each compatible α, ε may be determined as in Corollary 4.4,

$\delta(\alpha, \varepsilon)$:

$\alpha \backslash \varepsilon$	1	2	3	4	5	6	7	8
0	4	1	0	0	0	0	0	0
1	X	X	5	3	1	0	0	0
2	X	X	X	X	X	5	5	5

Here a crossed-out cell in the array means that α and ε are incompatible. Since the minimal α-width $\breve{\varepsilon}(\alpha)$ is the first ε compatible with α, circled entries in the array pick out $\breve{\varepsilon}(1) = 3$, $\breve{\varepsilon}(2) = 6$. In terms of the matrix $\breve{A}$, $\breve{\varepsilon}(\alpha)$ can be read off by looking at its last row: the α-th 1 of this row occurs in column $\breve{\varepsilon}(\alpha)$.

We conclude this section by listing some properties of $\delta(\alpha, \varepsilon)$:

(4.22) $\qquad \delta(\alpha, \varepsilon) \geq (\alpha, \varepsilon + 1),$

(4.23) $\qquad \delta(\alpha - 1, \tilde{\varepsilon}(\alpha)) = 0,$

(4.24) $\qquad \delta(\alpha, \tilde{\varepsilon}(\alpha)) > 0,$

(4.25) $\qquad \delta(\alpha, \tilde{\varepsilon}(\alpha)) = \delta(\alpha - 1, \breve{\varepsilon}(\alpha) - 1) + m - s_{\tilde{\varepsilon}(\alpha)}.$

The first three of these are evident, either from the definition of multiplicity or from the multiplicity matrix $\tilde{M}$. The last is easily proved using $\tilde{M}$. The property (4.25) can also be established from the formula (3.13) for $\delta(\alpha, \varepsilon)$, but this approach is more complicated.

5. THE SEQUENCES $\mu_A(\beta)$ AND $\bar{\mu}(\beta)$.

Let $\mathfrak{A} = \mathfrak{A}(R, S)$ be a normalized class and suppose that β is an integer parameter in the range

$$(5.1) \qquad 1 \leq \beta \leq r_1.$$

For each A in $\mathfrak{A}$ let $\mu_A(\beta)$ denote the maximal number of columns of A all of whose rows sums are at most β. (For example, if A is the line-point incidence matrix of a projective plane, then $\mu_A(2)$ is the maximal number of points, no three of which are collinear, i.e., the size of a maximal oval in the plane.) In this section we point out the close connection between this concept and that of width. In particular, we show that the preceding discussion on multiplicity and minimal width solves the problem of determining the class sequence

$$(5.2) \qquad \bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta).$$

It will simplify matters in this section if we extend the range of β in (5.1) to include $\beta = 0$ by defining $\mu_A(0) = 0$. We also take $\varepsilon_A(0) = 0$.

By the complementary class $\mathfrak{A}' = \mathfrak{A}(R', S')$ of $\mathfrak{A} = \mathfrak{A}(R, S)$, we mean the class of all (0, 1)-matrices of size m by n with row-sum vector

$$(5.3) \qquad R' = (n - r_m, n - r_{m-1}, \ldots, n - r_1),$$

and column-sum vector

$$(5.4) \qquad S' = (m - s_n, m - s_{n-1}, \ldots, m - s_1).$$

For the purposes of this discussion we take $\mathfrak{A}$ so that $r_1 < n$, $s_1 < m$. This is no real restriction and makes the complementary class normalized. There is of course a natural correspondence between the matrices of $\mathfrak{A}$ and those of $\mathfrak{A}'$, given by taking the complement of A and reversing the order of its rows and columns. We denote the resulting matrix by A' and call it the <u>class complement</u> of A.

<u>Lemma 5.1</u>. <u>Let</u> α <u>and</u> β <u>be integers in the respective intervals</u>

$$(5.5) \qquad 0 \leq \alpha \leq n - r_1,$$

$$(5.6) \qquad 0 \leq \beta \leq r_1,$$

<u>and let</u> A' <u>be the class complement of</u> A. <u>Then</u>

$$(5.7) \qquad \varepsilon_{A'}(\alpha) \leq \alpha + \beta$$

<u>if and only if</u>

$$(5.8) \qquad \mu_A(\beta) \geq \alpha + \beta.$$

<u>Proof</u>. Note that $\alpha + \beta$ ranges over the interval $0 \leq \alpha + \beta \leq n$.

Assume (5.7). Then there are $\alpha + \beta$ columns of A' having

at least α 1's in each row. Hence there are $\alpha + \beta$ columns of A having at most $\beta = (\alpha + \beta) - \alpha$ 1's in each row. Thus (5.8) holds. Conversely, if (5.8) holds, so that A has $\alpha + \beta$ columns with at most β 1's in each row, then A' has $\alpha + \beta$ columns with at least α 1's in each row. Hence (5.7) holds.

We use this lemma a number of times in the proof of Theorem 5.2, which shows that the sequences $\varepsilon_{A'}(\alpha)$ and $\mu_A(\beta)$ determine each other.

Theorem 5.2. (i) Let α be fixed in the interval (5.5) and let β be the least integer in the interval (5.6) for which $\mu_A(\beta) - \beta \geq \alpha$. Then $\varepsilon_{A'}(\alpha) = \alpha + \beta$. Conversely, if α is fixed in the interval (5.5) and if $\varepsilon_{A'}(\alpha) = \alpha + \beta$, then β is the least integer in (5.6) for which $\mu_A(\beta) - \beta \geq \alpha$.

(ii) Let β be fixed in the interval (5.6) and let α be the largest integer in the interval (5.5) for which $\varepsilon_{A'}(\alpha) - \alpha \leq \beta$. Then $\mu_A(\beta) = \alpha + \beta$. Conversely, if β is fixed in the interval (5.6) and if $\mu_A(\beta) = \alpha + \beta$, then α is the largest integer in (5.5) for which $\varepsilon_{A'}(\alpha) - \alpha \leq \beta$.

Proof. Observe that both sequences $\varepsilon_{A'}(\alpha) - \alpha$ and $\mu_A(\beta) - \beta$ are monotone nondecreasing. We now prove (i). Let α be fixed in (5.5) and let β be the least integer in (5.6) for which $\mu_A(\beta) - \beta \geq \alpha$. Such a β exists, since $\mu_A(r_1) - r_1 = n - r_1 \geq \alpha$. By Lemma 5.1, we have $\varepsilon_{A'}(\alpha) \leq \alpha + \beta$. If $\beta = 0$, then $\alpha = 0$, and the conclusion follows. If $\beta > 0$, then $\mu_A(\beta - 1) - (\beta - 1) < \alpha$. Hence by Lemma 5.1,

$\varepsilon_{A'}(\alpha) > \alpha + \beta - 1$. Thus $\alpha + \beta - 1 < \varepsilon_{A'}(\alpha) \leq \alpha + \beta$, and hence $\varepsilon_{A'}(\alpha) = \alpha + \beta$.

Conversely, suppose $\varepsilon_{A'}(\alpha) = \alpha + \beta$. We show first that $0 \leq \beta \leq r_1$. Clearly $0 \leq \beta$. Since also $\varepsilon_{A'}(n - r_1) \leq n$, then $\beta = \varepsilon_{A'}(\alpha) - \alpha \leq \varepsilon_{A'}(n - r_1) - (n - r_1) \leq r_1$. Now by hypothesis and Lemma 5.1, we have $\mu_A(\beta) - \beta \geq \alpha$. If also $\mu_A(\beta - 1) - (\beta - 1) \geq \alpha$, Lemma 5.1 implies $\varepsilon_{A'}(\alpha) \leq \alpha + \beta - 1$, a contradiction. Hence β is the least integer in (5.6) for which $\mu_A(\beta) - \beta \geq \alpha$.

The proof of (ii) is similar.

Let $\mathfrak{A}$ be a normalized class and let the complementary normalized class $\mathfrak{A}'$ have minimal width sequence $\tilde{\varepsilon}(\alpha)$. The discussion of the preceding section shows that the matrix $\tilde{A}$ in $\mathfrak{A}'$ has width $\tilde{\varepsilon}(\alpha)$ for each $\alpha = 0, 1, \ldots, n - r_1$. It follows that the matrix $\tilde{A}'$ in $\mathfrak{A}$ yields the sequence $\bar{\mu}(\beta)$:

$$(5.9) \qquad \bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta) = \mu_{\tilde{A}'}(\beta), \qquad \beta = 0, 1, \ldots, r_1.$$

The sequences $\tilde{\varepsilon}(\alpha)$ for $\mathfrak{A}'$ and $\bar{\mu}(\beta)$ for $\mathfrak{A}$ determine each other in the manner outlined in Theorem 5.2. In terms of the matrix $\tilde{A}$ in $\mathfrak{A}'$, the integer $\bar{\mu}(\beta)$ for $\mathfrak{A}$ can be singled out as follows: if the $(\beta + 1)$-st 0 of the last row of $\tilde{A}$ occurs in column j, then $\bar{\mu}(\beta) = j - 1$.

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